

Mirror Kähler potential on Calabi–Yau 3-folds *

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Abstract

The classical Kähler potential is a real-valued function (KP) such that one can determine a Kähler (symplectic) structure by differentiating KP. We define a mirror Kähler potential on Calabi–Yau 3-folds, a real-valued function (MKP) such that one can determine a complex structure by differentiating MKP.

KEYWORDS: special holonomy, Kähler potential, Calabi–Yau manifold

1 Introduction

The Kähler potential is a real-valued function φ such that $dd^c\varphi$ is the Kähler (symplectic) form. We propose a mirror construction in complex dimension 3, namely define a real-valued function φ such that φ acted by the certain symplectic differential operator is a 3-form that entirely determines a complex structure. The 3-forms that determine a complex structure are described by Hitchin in [1, 2].

In [3] we propose a new equation analogous to the complex Monge–Ampère equation studied by Calabi and Yau [4, 5]. New equation describes deformation of the complex structure, whereas the classical case describes deformation of the Kähler or symplectic one. It turns out that the mirror Kähler potential is a solution of the new equation.

In paper [6] the generalized Monge–Ampère equation and the Kähler potential are defined. However, for the case of the Kähler geometry they reduce to the usual Monge–Ampère equation and Kähler potential respectively.

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2 Background

First, note that hereafter we use smooth objects only: functions, manifolds etc.

2.1 The Kähler potential

Let us briefly recall the definition of the Kähler potential and the dd^c -lemma.

Let M be a Kähler manifold. The following statement called dd^c -lemma holds on M

$$\ker d \cap \operatorname{im} d^c = \operatorname{im} d \cap \ker d^c = \operatorname{im} dd^c.$$

The dd^c -lemma implies existence of the Kähler potential, a real-valued function φ such that $dd^c = \omega$, where ω is a Kähler structure. If manifold is compact, then the Kähler form can not be exact and potential can not be a globally defined function. However, it can be still defined locally.

2.2 Symplectic Hodge theory

In this subsection we define a symplectic differential operator analogous to d^c . Let (M, ω) be a symplectic manifold of real dimension $2n$. By $*_s$ denote a symplectic Hodge star. The action of $*_s$ on the k -forms is uniquely determined by the following formula:

$$\alpha \wedge *_s \beta = (\omega^{-1})^k(\alpha, \beta) \frac{\omega^n}{n!}.$$

By definition, put

$$d^s = (-1)^{k+1} *_s d *_s. \tag{1}$$

Note that d^s decreases degree of the form by one and $dd^s = -d^s d$. The following statement called dd^s -lemma holds on Kähler manifolds.

$$\ker d \cap \operatorname{im} d^s = \operatorname{im} d \cap \ker d^s = \operatorname{im} dd^s.$$

Let us note that the notation of d^s operator is proposed in this paper. In literature it has various notations: δ , d^Λ , $d^\mathcal{J}$ etc. We use current notation due to its symmetry with the notation of the d^c operator.

For further information on the symplectic Hodge theory see for example [7] and references therein.

2.3 Stable forms

In this subsection we recall the definition of a stable forms by Hitchin [1, 2]. Let V be a real m -space. A real form $\rho \in \Lambda^p V^*$ is called stable iff the orbit of ρ under the natural action of $GL(V)$ is open. This definition can be ported in the clear way to the case of forms on manifolds.

If $p = 2$ and $m = 2n$, then stability is equivalent to the non-degeneracy condition of symplectic form. If $p = 3$ and $m = 6$, then any stable form is a real or imaginary part of some holomorphic volume form.

Here we list some properties of stable 3-forms.

- Stability can be established by some algebraic criterion on 3-form.
- Any stable *real* 3-form completely determines some almost complex structure on V .
- For any stable form ρ there exists a dual stable forms $\hat{\rho}$ such that $\rho + i\hat{\rho}$ is a $SL(3, \mathbb{C})$ -invariant form. If $\rho + i\hat{\rho}$ is closed, then almost complex structure determined by ρ is integrable and $\rho + i\hat{\rho}$ is a holomorphic volume form with respect to the complex structure determined by ρ .
- For any stable form ρ there exists a frame $\{e^i\}$ of V^* such that

$$\rho = e^{135} - e^{245} - e^{146} - e^{236}, \quad \hat{\rho} = e^{235} + e^{145} + e^{136} - e^{246}, \quad (2)$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$.

3 The mirror Kähler potential

3.1 Definition

Let M be a compact Calabi–Yau 3-fold with Kähler form ω and holomorphic volume form $\Omega = \rho + \sqrt{-1}\sigma$. By Darboux’ theorem, for any point $p \in M$

there exists a local co-ordinate chart $\{x^i\}$ in the neighborhood of p such that locally

$$\omega = dx^{12} + dx^{34} + dx^{56}, \quad (3)$$

where $dx^{ij} = dx^i \wedge dx^j$. Put

$$\rho_0 = dx^{135} - dx^{245} - dx^{146} - dx^{236}, \quad \sigma_0 = dx^{235} + dx^{145} + dx^{136} - dx^{246}, \quad (4)$$

or

$$\Omega_0 = \rho_0 + \sqrt{-1}\sigma_0 = (dx^1 + \sqrt{-1}dx^2) \wedge (dx^3 + \sqrt{-1}dx^4) \wedge (dx^5 + \sqrt{-1}dx^6). \quad (5)$$

Definition 1. We say that a locally defined real-valued function φ is a *local mirror Kähler potential* iff

$$\rho = dd^s \varphi \sigma_0, \quad \sigma = -dd^s \varphi \rho_0$$

or

$$\Omega = -\sqrt{-1}dd^s \varphi \Omega_0,$$

where $\Omega_0 = \rho_0 + \sqrt{-1}\sigma_0$.

Since stable forms completely determine complex structure, one can see that the mirror Kähler potential determines complex structure whenever the symplectic one is given.

As in the classical case one can not define the Kähler potential globally unless the holomorphic volume form is exact. Nevertheless, we give the following definition.

Definition 2. We say that a global function on M is a *global mirror Kähler potential* iff for $\Omega' \sim \Omega$

$$\Omega' = \Omega - \sqrt{-1}dd^s \varphi \Omega.$$

Example. Suppose \mathbb{C}^3 with a flat Hermitian metric $\sum dz^i \otimes d\bar{z}^i$; the Kähler form $\omega = (\sqrt{-1}/2) \sum dz^i \wedge d\bar{z}^i$ and holomorphic volume form $\Omega = dz^1 \wedge dz^2 \wedge dz^3$. Then the following identities hold:

$$\omega = dd^c \varphi, \quad \Omega = -(\sqrt{-1}/3)dd^s \varphi \Omega,$$

where $\varphi = \sum |z^i|^2$.

3.2 Does the mirror Kähler potential exist?

The question about existence of the local or global mirror Kähler potential is open. The possible line of attack could be using the classical continuity method. Consider for example \mathbb{R}^6 with the standard Euclidean symplectic structure ω_0 and compatible complex structure given by $\Omega = \rho + \sqrt{-1}\sigma$. We have seen in the example above that there exists a mirror Kähler potential whenever there is a flat metric, i.e., $\rho = \rho_0$; $\sigma = \sigma_0$.

Now consider a parametrized system of equations:

$$dd^s\varphi_t\rho_0 = -(1-t)\sigma_0 - t\sigma; \quad dd^s\varphi_t\sigma_0 = (1-t)\rho_0 + t\rho; \quad t \in [0, 1]. \quad (6)$$

Since the set of almost complex structures compatible with the given symplectic form is convex, RHS describes the continuous family of stable 3-forms.

Then we can use the continuity method. Let \mathcal{T} be a subset of $[0, 1]$ such that $t \in \mathcal{T}$, whenever parametrized equation has solution for that t . Since $0 \in \mathcal{T}$, it is not empty. Now one has to establish some estimates on the equation (6) and prove that \mathcal{T} is open and closed in $[0, 1]$.

We conjecture that (at least) for compact Calabi–Yau 3-folds there exist local and global mirror Kähler potentials.

Conjecture 1. *Let M be a compact Calabi–Yau 3-fold with Kähler form ω and holomorphic volume form $\Omega = \rho + \sqrt{-1}\sigma$. Then for any point $p \in M$ there locally exists a co-ordinate chart $\{x^i\}$ such that ω takes form (3) and a real-valued function φ such that*

$$\Omega = -\sqrt{-1}dd^s\varphi\Omega_0,$$

where Ω_0 is defined by the formula (5) with respect to the local chart.

Conjecture 2. *Let M be a compact Calabi–Yau 3-fold with Kähler form ω and holomorphic volume form $\Omega = \rho + \sqrt{-1}\sigma$. Then for $\Omega' \sim \Omega$ there exists a global real-valued function φ such that*

$$\Omega' = \Omega - \sqrt{-1}dd^s\varphi\Omega.$$

3.3 Functional class

Let us recall the definition of the plurisubharmonic functions. Assume \mathbb{C}^3 with a symplectic form ω . Let $\mathcal{C}(\omega)$ be a convex conical set of alternating 2-vectors such that for any $\xi \in \mathcal{C}(\omega)$

$$\omega(\xi) > 0.$$

Recall that a real-valued (smooth) function φ is called plurisubharmonic iff for any $\xi \in \mathcal{C}(\omega)$

$$(dd^c\varphi)(\xi) \geq 0.$$

Function φ is called pluriharmonic iff φ and $-\varphi$ are plurisubharmonic functions.

Now let us state analogous definition concerning special Lagrangian submanifolds. Let $\rho + \sqrt{-1}\sigma$ be a holomorphic volume form on \mathbb{C}^3 ; then define sets of 3-vectors $\mathcal{C}(\rho)$ and $\mathcal{C}(\sigma)$ as above.

Definition 3. We call a stable 3-form τ positive (strictly positive) modulo stable form ρ and denote by $\tau \geq 0$ ($\tau > 0$) mod ρ iff for any $\xi \in \mathcal{C}(\rho)$

$$\tau(\xi) \geq 0 \quad (\tau(\xi) > 0).$$

We call a stable 3-form τ negative (strictly negative) modulo stable form ρ and denote by $\tau \leq 0$ ($\tau < 0$) mod ρ iff $-\tau \geq 0$ ($-\tau > 0$) mod ρ .

Definition 4. A real-valued function φ is called *special Lagrangian plurisubharmonic* iff for any $\xi \in \mathcal{C}(\rho)$ and $\eta \in \mathcal{C}(\sigma)$

$$(dd^s\varphi\sigma)(\xi) \geq 0; \quad (dd^s\varphi\rho)(\eta) \leq 0, \tag{7}$$

or equivalently

$$dd^s\varphi\sigma \geq 0 \quad \text{mod } \rho; \quad dd^s\varphi\rho \leq 0 \quad \text{mod } \sigma. \tag{8}$$

Definition 5. A real-valued function φ is called *strictly special Lagrangian plurisubharmonic* iff inequalities (7) or equivalently (8) are strict.

Remark. This definition is closely related with the one given in [8]. In fact, these two definitions determine the same class of functions. However, they use a differential operator d^ϕ related with the calibration ϕ . The d^ϕ is first defined in [9] for the case of G_2 -manifolds.

Claim. *Any special Lagrangian plurisubharmonic function is plurisubharmonic.*

Proof. By Hörmander's definition of G -subharmonic functions, where G is a linear group [10, Definition 5.1.1] special Lagrangian plurisubharmonic functions are $Sp(3, \mathbb{C})$ -subharmonic. Namely, the set \mathcal{S} of special Lagrangian plurisubharmonic functions satisfies the following conditions.

- \mathcal{S} contains every affine function;
- \mathcal{S} is invariant under $Sp(3, \mathbb{C})$;
- the weak maximum principle is valid for \mathcal{S} ;
- \mathcal{S} is maximal with the preceding properties.

By [10, Theorem 5.1.7], the set of $Sp(3, \mathbb{C})$ -subharmonic functions is a set of plurisubharmonic functions. □

Corollary. *The mirror Kähler potential is a plurisubharmonic function.*

4 New equation

In paper [3] we propose a new equation on the 3-dimensional Calabi–Yau metrics and prove the following solution existence theorem.

Theorem. *Let (M, ω) be a compact Kähler 3-manifold such that $c_1(M) = 0$; let $\Omega = \rho + i\sigma$ be a holomorphic volume form on M . Then the following equation on the unknown real 3-forms α and β*

$$(\rho + dd^s\alpha) \wedge (\sigma + dd^s\beta) = e^F \rho \wedge \sigma \quad (9)$$

or equivalently

$$(\Omega + dd^s\psi) \wedge (\bar{\Omega} + dd^s\bar{\psi}) = e^F \Omega \wedge \bar{\Omega}, \quad \psi = \alpha + \sqrt{-1}\beta \quad (10)$$

has solution provided that

1. $\rho + dd^s\alpha$ and $\sigma + dd^s\beta$ are primitive stable forms;
2. $\sigma + dd^s\beta$ is dual to $\rho + dd^s\alpha$ in the sense of the stable forms;
3. real function F is normalized: $\int_M e^F \rho \wedge \sigma = \int_M \rho \wedge \sigma$.

If the global mirror Kähler potential φ exists, then equations (9) and (10) take form

$$(\rho + dd^s\varphi\sigma) \wedge (\sigma - dd^s\varphi\rho) = e^F \rho \wedge \sigma \quad (11)$$

and

$$(\Omega - \sqrt{-1}dd^s\varphi\Omega) \wedge (\bar{\Omega} + \sqrt{-1}dd^s\varphi\bar{\Omega}) = e^F \Omega \wedge \bar{\Omega} \quad (12)$$

respectively.

Conjecture 3. *Let M be a compact Calabi–Yau 3-fold with Kähler form ω and holomorphic volume form $\Omega = \rho + \sqrt{-1}\sigma$. Then equation (11) or equivalently (12) has unique solution: a real-valued function φ provided that*

1. $\rho + dd^s\varphi\sigma > 0 \pmod{\rho}$ and $\sigma - dd^s\varphi\rho > 0 \pmod{\sigma}$, where positivity is in the sense of the Definition 3;
2. $\int_M e^F \omega^n = \int_M \omega^n$;
3. $\int_M \varphi \omega^n = 0$.

Obviously, this conjecture follows from the Conjecture 2.

Now let us write down the local form of the equation (11). First, recall the local form of the Monge–Ampère equation.

Suppose U is an open set of M with co-ordinate chart $\{z^i\}$ such that $\Omega = dz^1 \wedge dz^2 \wedge dz^3$ on U ; then the global equation $(\omega + dd^c\varphi)^n = e^F \omega^n$ is locally equivalent to the following equation on U

$$\det \varphi_{i\bar{j}} = \text{const},$$

where $\varphi_{i\bar{j}} = \partial^2 \varphi / \partial z^i \partial \bar{z}^j$.

Now suppose U is an open set of M with co-ordinate chart $\{x^i\}$ such that $\omega = dx^{12} + dx^{34} + dx^{56}$ on U ; then the global equation (11) is locally equivalent to the following equation on U

$$\begin{aligned} & (\varphi_{22} + \varphi_{33} + \varphi_{55})(\varphi_{11} + \varphi_{44} + \varphi_{66}) \\ & + (\varphi_{11} + \varphi_{44} + \varphi_{55})(\varphi_{22} + \varphi_{33} + \varphi_{66}) \\ & + (\varphi_{11} + \varphi_{33} + \varphi_{66})(\varphi_{22} + \varphi_{44} + \varphi_{55}) \\ & + (\varphi_{22} + \varphi_{44} + \varphi_{66})(\varphi_{11} + \varphi_{33} + \varphi_{55}) \\ & - (\varphi_{12} + \varphi_{34} + \varphi_{56})^2 - (-\varphi_{12} - \varphi_{34} + \varphi_{56})^2 - (\varphi_{12} - \varphi_{34} - \varphi_{56})^2 \\ & - (-\varphi_{12} + \varphi_{34} - \varphi_{56})^2 - 2[(\varphi_{13} - \varphi_{24})^2 + (\varphi_{36} + \varphi_{45})^2 + (\varphi_{15} - \varphi_{26})^2 \\ & + (\varphi_{16} + \varphi_{25})^2 + (\varphi_{35} - \varphi_{46})^2 + (\varphi_{14} + \varphi_{23})^2] = \text{const}, \end{aligned} \tag{13}$$

where $\varphi_{ij} = \partial^2 \varphi / \partial x^i \partial x^j$.

To investigate equation (13) consider the case when the Calabi–Yau manifold is fibered by flat special Lagrangian tori [11] and perform analysis similar to [12]. Suppose manifold is semi-flat and the mirror Kähler potential depends on the odd co-ordinates only. Then equation (13) takes the following form.

$$\varphi_{11}\varphi_{33} + \varphi_{11}\varphi_{55} + \varphi_{33}\varphi_{55} - \varphi_{13}^2 - \varphi_{15}^2 - \varphi_{35}^2 = \text{const}, \tag{14}$$

where $\varphi_{ij} = \partial^2 \varphi / \partial x^i \partial x^j$. Obviously, the equation (14) is the sum of three real Monge–Ampère equations. It is known that for any solution of the real Monge–Ampère equation one can produce a new solution by performing the Legendre transformation. Therefore, by performing the partial Legendre transformation on the mirror Kähler potential, we obtain new solutions of (13) just as in the classical case. Note that the partial Legendre transformation of a plurisubharmonic function is again plurisubharmonic [13].

Conjecturally, equation (13) is also related to the some type of the Monge–Ampère equations [14]. This is an open question.

5 Conclusion

In conclusion we state some open problems.

1) It is natural to consider higher dimensional cases, though the holomorphic volume form is not stable if $n > 3$. The uniqueness of the $n = 3$ case is that a single real 3-form determines complex structure. In higher dimensions one needs a pair of real n -forms to determine a complex structure.

2) What is a relation between the classical and mirror Kähler potentials? What is a relation between the complex Monge–Ampère equation and the new equation?

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